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A UNIFIED PRESENTATION OF BETA, GAUSS AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Abstract. In this article, we propose a new extension of beta function by making use of the Bessel-Struve kernel function. Here, first we derive some fundamental properties of this function and then we present a new extension of beta distribution in terms of our proposed beta function. Furthermore, by using the definition of our new beta function, we introduce and investigate a new generalization of Gauss and confluent hypergeometric functions.

Keywords: Beta function, extended beta function, Gauss hypergeometric function, extended Gauss hypergeometric function, confluent hypergeometric function, extended confluent hypergeometric function, Bessel-Struve kernel function.
MSC(2010): 33B15, 33B20, 33C05, 33C15.

1. Introduction

Throughout in this paper, let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of natural numbers, real numbers and complex numbers, respectively, and let

$$\mathbb{N} := \{1, 2, 3, ...\},\ \mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

The classical beta function $\mathbb{B}(a, b)$ is defined by

$$\mathbb{B}(a,b) = \int_0^1 u^{a-1}\ (1-u)^{b-1} du \tag{1.1}$$

$$(\Re(a) > 0,\ \Re(b) > 0).$$

Due to diverse applications of beta function in a wide range of engineering and sciences, numerous researchers have introduced and investigated several extensions of (1.1) (see, for example, [1], [2], [3], [6], [8], [9] and [12]).

Chaudhry et al. [3] introduced a useful generalization of (1.1) by

$$\mathbb{B}_\tau(a,b) = \int_0^1 u^{a-1}\ (1-u)^{b-1}\ \exp\left[-\frac{\tau}{u(1-u)}\right] du \tag{1.2}$$

$$(\Re(a) > 0,\ \Re(b) > 0,\ \Re(\tau) > 0).$$

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It is easily seen that for $\tau = 0$, (1.2) reduces to (1.1). By using (1.2), Chaudhry et al. [4] generalized the Gauss hypergeometric function and the confluent hypergeometric function, respectively, as follows:

$$F_\tau(a_1, a_2; a_3; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \, \mathbb{B}_\tau(a_2+n, a_3-a_2)}{\mathbb{B}(a_2, a_3-a_2)} \frac{z^n}{n!} \tag{1.3}$$
$$(\tau \geq 0,\ |z| < 1,\ \Re(a_3) > \Re(a_2) > 0)$$

and

$$\Phi_\tau(a_2; a_3; z) = \sum_{n=0}^{\infty} \frac{\mathbb{B}_\tau(a_2+n, a_3-a_2)}{\mathbb{B}(a_2, a_3-a_2)} \frac{z^n}{n!} \tag{1.4}$$
$$(\tau \geq 0,\ \Re(a_3) > \Re(a_2) > 0).$$

Among several interesting results given in [4], the following Euler's type integral representations are recalled:

$$F_\tau(a_1, a_2; a_3; z) = \frac{1}{\mathbb{B}(a_2, a_3-a_2)} \tag{1.5}$$
$$\times \int_0^1 u^{a_2-1} \, (1-u)^{a_3-a_2-1} \, (1-zu)^{-a_1} \exp\left[-\frac{\tau}{u(1-u)}\right] du$$
$$(\tau \geq 0,\ |\arg(1-z)| < \pi,\ \Re(a_3) > \Re(a_2) > 0)$$

and

$$\Phi_\tau(a_2; a_3; z) = \frac{1}{\mathbb{B}(a_2, a_3-a_2)} \tag{1.6}$$
$$\times \int_0^1 u^{a_2-1} \, (1-u)^{a_3-a_2-1} \, \exp\left[zu - \frac{\tau}{u(1-u)}\right] du$$
$$(\tau \geq 0,\ \Re(a_3) > \Re(a_2) > 0).$$

The main motive of this paper is to introduce a further extension of beta function by making use of the Bessel-Struve kernel function and also to present a new generalization of Gauss and confluent hypergeometric functions.

The Bessel-Struve kernel function $S_k(\lambda t), \lambda \in \mathbb{C}$ which is unique solution of the initial value problem $\ell_k u(t) = \lambda^2 u(t)$ with the initial condition $u(0) = 1$ and $u'(0) = \frac{\lambda \Gamma(k+1)}{\sqrt{\pi}\, \Gamma(k+\frac{3}{2})}$ is given by (see, [5], see also [7])

$$S_k(\lambda t) = j_k(i\lambda t) - i h_k(i\lambda t), \forall\ t \in \mathbb{C},$$

where j_k and h_k are the normalized Bessel and Struve functions. The series representation, of the Bessel-Struve kernel function is given as follows:

$$S_k(\lambda t) = \frac{\Gamma(k+1)}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(\lambda t)^n \Gamma(\frac{n+1}{2})}{n! \; \Gamma(\frac{n}{2}+k+1)}. \tag{1.7}$$

Also, we have the following relations of Bessel-Struve kernel function with the exponential functions, modified Bessel function and Struve function (see, [5], see also [7]):

$$S_{-\frac{1}{2}}(t) = e^t, \tag{1.8}$$

$$S_{\frac{1}{2}}(t) = \frac{e^t - 1}{t}, \tag{1.9}$$

$$S_0(t) = I_0(t) + L_0(t) \tag{1.10}$$

and

$$S_1(t) = \frac{2I_1(t) + L_1(t)}{t}. \tag{1.11}$$

where I_0, L_0 and I_1, L_1 are the modified Bessel and Struve functions of order zero and one respectively (see, [11], see also [10]).

2. Extended beta function

In this section, we define the following extension of beta function by making use of the Bessel-Struve kernel function $S_k(\lambda t)$ given in (1.7):

$$\mathbb{B}^{(\tau,k)}(a,b) = \int_0^1 u^{a-1}\,(1-u)^{b-1}\,S_k\left[-\frac{\tau}{u(1-u)}\right]du \tag{2.1}$$

$$(\Re(a) > 0,\ \Re(b) > 0,\ \Re(k) > -1,\ \tau \geq 0).$$

If we set $k = -\frac{1}{2}$ then (2.1) reduces to (1.2), which further for $\tau = 0$ gives the classical beta function (1.1).
Thus, we have

$$\mathbb{B}^{(\tau,-\frac{1}{2})}(a,b) = \mathbb{B}_\tau(a,b);\ \mathbb{B}_0(a,b) = \mathbb{B}(a,b). \tag{2.2}$$

Further, if we set $k = \frac{1}{2}$ in (2.1) and then by using the fact $S_{\frac{1}{2}}(t) = \frac{e^t-1}{t}$, we get

$$\mathbb{B}^{(\tau,\frac{1}{2})}(a,b) = \frac{1}{\tau}\left[\mathbb{B}(a+1,b+1) - \mathbb{B}_\tau(a+1,b+1)\right], \tag{2.3}$$

where $\mathbb{B}(a+1,b+1)$ and $\mathbb{B}_\tau(a+1,b+1)$ are the classical and extended beta function defined by (1.1) and (1.2), respectively.

Moreover, for $k = 0$, (2.1) yields

$$\mathbb{B}^{(\tau,0)}(a,b) = \int_0^1 u^{a-1}\,(1-u)^{b-1}\left\{I_0\left[\frac{-\tau}{u(1-u)}\right] + L_0\left[\frac{-\tau}{u(1-u)}\right]\right\}du, \tag{2.4}$$

and for $k = 1$, (2.1) can be expressed as

$$\mathbb{B}^{(\tau,1)}(a,b) = -\frac{1}{\tau}\int_0^1 u^{a}\,(1-u)^{b}\left\{2I_1\left[\frac{-\tau}{u(1-u)}\right] + L_1\left[\frac{-\tau}{u(1-u)}\right]\right\}du. \tag{2.5}$$

Integral representation of $\mathbb{B}^{(\tau,k)}(a,b)$

Theorem 2.1. *For $\Re(k) > -1$ and $\tau \geq 0$, we have the following integral representations of $\mathbb{B}^{(\tau,k)}(a,b)$:*

$$\mathbb{B}^{(\tau,k)}(a,b) = 2\int_0^{\frac{\pi}{2}} \cos^{2a-1} t\ \sin^{2b-1} t\ S_k(-\tau \sec^2 t\ \csc^2 t)dt; \tag{2.6}$$

$$\mathbb{B}^{(\tau,k)}(a,b) = \int_0^{\infty} \frac{w^{a-1}}{(1+w)^{a+b}}\ S_k\left[-\tau\left(2+w+\frac{1}{w}\right)\right] dw; \tag{2.7}$$

$$\mathbb{B}^{(\tau,k)}(a,b) = 2^{1-a-b}\int_{-1}^{1} (1+w)^{a-1}\ (1-w)^{b-1}\ S_k\left[-\frac{4\tau}{(1-w^2)}\right] dw. \tag{2.8}$$

Proof. On putting $u = \cos^2 t$, $u = \frac{w}{1+w}$ and $u = \frac{1+w}{2}$ in (2.1) gives, respectively, the integral representations (2.6)–(2.8). □

3. Properties of extended beta function

This section deals with some basic properties of our introduced beta function $\mathbb{B}^{(\tau,k)}(a,b)$.

Theorem 3.1. *The following result holds true for extended beta function $\mathbb{B}^{(\tau,k)}(a,b)$:*

$$\mathbb{B}^{(\tau,k)}(a,b) = \sum_{s=0}^{l} \binom{l}{s} \mathbb{B}^{(\tau,k)}(a+s, b+l-s),\ \text{where } l \in \mathbb{N}_0. \tag{3.1}$$

Proof. From (2.1), we have

$$\mathbb{B}^{(\tau,k)}(a,b) = \int_0^1 u^{a-1}\ (1-u)^{b-1}\{u+(1-u)\}\ S_k\left[-\frac{\tau}{u(1-u)}\right] du$$

$$\mathbb{B}^{(\tau,k)}(a,b) = \mathbb{B}^{(\tau,k)}(a+1,b) + \mathbb{B}^{(\tau,k)}(a,b+1). \tag{3.2}$$

Further, applying the same argument in the right side of (3.2), we get

$$\mathbb{B}^{(\tau,k)}(a,b) = \mathbb{B}^{(\tau,k)}(a+2,b) + 2\mathbb{B}^{(\tau,k)}(a+1,b+1) + \mathbb{B}^{(\tau,k)}(a,b+2),$$

continuing this process, by induction, we obtain the desired result. □

Theorem 3.2. *The following result holds true for extended beta function $\mathbb{B}^{(\tau,k)}(a,b)$:*

$$\mathbb{B}^{(\tau,k)}(a,1-b) = \sum_{s=0}^{\infty} \frac{(b)_s}{s!} \mathbb{B}^{(\tau,k)}(a+s,1). \tag{3.3}$$

Proof. On using (2.1) in the left side of (3.3), we get

$$\mathbb{B}^{(\tau,k)}(a,1-b) = \int_0^1 u^{a-1}\ (1-u)^{-b}\ S_k\left[-\frac{\tau}{u(1-u)}\right] du$$

$$= \int_0^1 u^{a-1} \sum_{s=0}^{\infty} \frac{(b)_s \ u^s}{s!} \ S_k \left[-\frac{\tau}{u(1-u)} \right] du.$$

Now interchanging the order of integration and summation in the above expression and then upon using (2.1), we arrive at our needed result. □

Theorem 3.3. *The following result holds true for extended beta function* $\mathbb{B}^{(\tau,k)}(a,b)$*:*

$$\mathbb{B}^{(\tau,k)}(a,b) = \sum_{s=0}^{\infty} \mathbb{B}^{(\tau,k)}(a+s, b+1). \tag{3.4}$$

Proof. The above theorem can be established with the help of (2.1) by writing $(1-u)^{b-1} = (1-u)^b \sum_{s=0}^{\infty} u^s$. We omit the details. □

4. The extended beta distribution

In this section, we consider the potential use of our extended beta function by defining the following new extended beta distribution:

$$f(u) = \begin{cases} \frac{1}{\mathbb{B}^{(\tau,k)}(a,b)} \ u^{a-1} \ (1-u)^{b-1} \ S_k \left[-\frac{\tau}{u(1-u)} \right] & (0 < u < 1) \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

$$(a, \ b \in \mathbb{R}, \ \tau \geq 0, \ \Re(k) > -1).$$

Next, we have discussed here some fundamental properties of our extended beta distribution (4.1).

For $n \in \mathbb{R}$, the n^{th} moment of U is given by

$$E(U^n) = \frac{\mathbb{B}^{(\tau,k)}(a+n, b)}{\mathbb{B}^{(\tau,k)}(a,b)} \tag{4.2}$$

$$(a, \ b \in \mathbb{R}, \ \tau \geq 0, \ \Re(k) > -1).$$

The particular case of (4.2) for $n = 1$ yields the mean of our extended beta distribution

$$E(U) = \frac{\mathbb{B}^{(\tau,k)}(a+1, b)}{\mathbb{B}^{(\tau,k)}(a,b)}. \tag{4.3}$$

The variance of our introduced distribution is defined by

$$Var(U) = E(U^2) - [E(U)]^2$$

$$Var(U) = \frac{\mathbb{B}^{(\tau,k)}(a+2, b) \ \mathbb{B}^{(\tau,k)}(a,b) - [\mathbb{B}^{(\tau,k)}(a+1,b)]^2}{[\mathbb{B}^{(\tau,k)}(a,b)]^2}. \tag{4.4}$$

The coefficient of variation of this distribution (which is defined as the ratio of the standard deviation and mean) can be expressed as

$$C.V = \sqrt{\frac{\mathbb{B}^{(\tau,k)}(a+2, b) \ \mathbb{B}^{(\tau,k)}(a,b)}{[\mathbb{B}^{(\tau,k)}(a+1,b)]^2} - 1}. \tag{4.5}$$

The moment generating function (m.g.f) about origin of this distribution is given by

$$M_U(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(U^r)$$

$$M_U(t) = \frac{1}{\mathbb{B}^{(\tau,k)}(a,b)} \sum_{r=0}^{\infty} \mathbb{B}^{(\tau,k)}(a+r,b) \frac{t^r}{r!}. \tag{4.6}$$

The characteristic function of the proposed distribution can be calculated as follows:

$$E(e^{itu}) = \sum_{r=0}^{\infty} \frac{i^r t^r}{r!} E(U^r)$$

$$E(e^{itu}) = \frac{1}{\mathbb{B}^{(\tau,k)}(a,b)} \sum_{r=0}^{\infty} \mathbb{B}^{(\tau,k)}(a+r,b) \frac{i^r t^r}{r!}. \tag{4.7}$$

The cumulative distribution function of our extended beta distribution (4.1) can be expressed as

$$F(u) = P[U < u] = \int_0^u f(u) du$$

$$F(u) = \frac{\mathbb{B}^{(\tau,k,u)}(a,b)}{\mathbb{B}^{(\tau,k)}(a,b)}, \tag{4.8}$$

where $\mathbb{B}^{(\tau,k,u)}(a,b)$ denotes the incomplete extended beta function defined by

$$\mathbb{B}^{(\tau,k,u)}(a,b) = \int_0^u u^{a-1}\, (1-u)^{b-1}\, S_k \left[-\frac{\tau}{u(1-u)} \right] du.$$

The reliability function (which is simply the complement of the cumulative distribution function) of our proposed distribution is given by

$$R(u) = P[U \geq u] = 1 - F(u) = \int_u^{\infty} f(u) du$$

$$R(u) = \frac{\mathbb{B}^{(\tau,k,u)}(a,b)}{\mathbb{B}^{(\tau,k)}(a,b)}, \tag{4.9}$$

where $\mathbb{B}^{(\tau,k,u)}(a,b)$ is the incomplete extended beta function defined by

$$\mathbb{B}^{(\tau,k,u)}(a,b) = \int_u^{\infty} u^{a-1}\, (1-u)^{b-1}\, S_k \left[-\frac{\tau}{u(1-u)} \right] du.$$

5. **Extended hypergeometric functions and their associated properties**

In this section, we present the following extensions of Gauss and confluent hypergeometric functions by making use of our extended beta function $\mathbb{B}^{(\tau,k)}(a,b)$:

$$F^{(\tau,k)}(a_1, a_2; a_3; t) = \sum_{l=0}^{\infty} \frac{(a_1)_l\ \mathbb{B}^{(\tau,k)}(a_2+l, a_3-a_2)}{\mathbb{B}(a_2, a_3-a_2)} \frac{t^l}{l!} \tag{5.1}$$

$$(\tau \geq 0,\ |t| < 1,\ \Re(a_3) > \Re(a_2) > 0,\ \Re(k) > -1)$$

and

$$\Phi^{(\tau,k)}(a_2;a_3;t)=\sum_{l=0}^{\infty}\frac{\mathbb{B}^{(\tau,k)}(a_2+l,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}\frac{t^l}{l!} \tag{5.2}$$

$$(\tau\geq 0,\ \Re(a_3)>\Re(a_2)>0,\ \Re(k)>-1).$$

Remark 5.1. For $k=-\frac{1}{2}$, (5.1) and (5.2) reduces to the known extensions of Gauss and confluent hypergeometric functions defined by Chaudhry et al. [4].

Theorem 5.2. *The following integral representations of our extended Gauss and confluent hypergeometric functions holds true:*

$$\begin{aligned}F^{(\tau,k)}(a_1,a_2;a_3;t)&=\frac{1}{\mathbb{B}(a_2,a_3-a_2)}\\ &\times\int_0^1 u^{a_2-1}\ (1-u)^{a_3-a_2-1}\ (1-tu)^{-a_1}S_k\left[-\frac{\tau}{u(1-u)}\right]du\end{aligned} \tag{5.3}$$

$$(\tau\geq 0,\ |\arg(1-t)|<\pi,\ \Re(a_3)>\Re(a_2)>0,\ \Re(k)>-1)$$

and

$$\Phi^{(\tau,k)}(a_2;a_3;t)=\frac{1}{\mathbb{B}(a_2,a_3-a_2)}\int_0^1 u^{a_2-1}\ (1-u)^{a_3-a_2-1}\ e^{tu}S_k\left[-\frac{\tau}{u(1-u)}\right]du \tag{5.4}$$

$$(\tau\geq 0,\ \Re(a_3)>\Re(a_2)>0,\ \Re(k)>-1).$$

Proof. Each of the above integral representations can be easily established by using the integral representation of our extended beta function (2.1) in the right sides of (5.1) and (5.2), respectively. □

Theorem 5.3. *The following differential formulas for our extended Gauss and confluent hypergeometric functions holds true:*

$$\frac{d^m}{dt^m}\left\{F^{(\tau,k)}(a_1,a_2;a_3;t)\right\}=\frac{(a_1)_m(a_2)_m}{(a_3)_m}F^{(\tau,k)}(a_1+m,a_2+m;a_3+m;t) \tag{5.5}$$

$$(\tau\geq 0,\ \Re(k)>-1,\ m\in\mathbb{N}_0)$$

and

$$\frac{d^m}{dt^m}\left\{\Phi^{(\tau,k)}(a_2;a_3;t)\right\}=\frac{(a_2)_m}{(a_3)_m}\Phi^{(\tau,k)}(a_2+m;a_3+m;t) \tag{5.6}$$

$$(\tau\geq 0,\ \Re(k)>-1,\ m\in\mathbb{N}_0).$$

Proof. On differentiating (5.1) with respect to t, we get

$$\frac{d}{dt}\left\{F^{(\tau,k)}(a_1,a_2;a_3;t)\right\}=\sum_{l=1}^{\infty}\frac{(a_1)_l\ \mathbb{B}^{(\tau,k)}(a_2+l,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}\frac{t^{l-1}}{(l-1)!}.$$

On replacing l by $l+1$, we have

$$\frac{d}{dt}\left\{F^{(\tau,k)}(a_1,a_2;a_3;t)\right\}=\sum_{l=0}^{\infty}\frac{(a_1)_{l+1}\ \mathbb{B}^{(\tau,k)}(a_2+l+1,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}\frac{t^l}{l!}.$$

Now by using $\mathbb{B}(a_2, a_3 - a_2) = \frac{a_3}{a_2}\mathbb{B}(a_2 + 1, a_3 - a_2)$ and $(a_1)_{l+1} = a_1(a_1 + 1)_l$, we get

(5.7)
$$\frac{d}{dt}\left\{F^{(\tau,k)}(a_1, a_2; a_3; t)\right\} = \frac{a_1 a_2}{a_3}\sum_{l=0}^{\infty}\frac{(a_1 + 1)_l\ \mathbb{B}^{(\tau,k)}(a_2 + l + 1, a_3 - a_2)}{\mathbb{B}(a_2 + 1, a_3 - a_2)}\frac{t^l}{l!}.$$

$$= \frac{a_1 a_2}{a_3}F^{(\tau,k)}(a_1 + 1, a_2 + 1; a_3 + 1; t).$$

Again differentiating (5.7) with respect to t, we get

$$\frac{d^2}{dt^2}\left\{F^{(\tau,k)}(a_1, a_2; a_3; t)\right\} = \frac{a_1(a_1 + 1)a_2(a_2 + 1)}{a_3(a_3 + 1)}F^{(\tau,k)}(a_1 + 2, a_2 + 2; a_3 + 2; t).$$

Continuing this process, by induction we obtain the required result (5.5). Similarly we can establish the result (5.6). □

Theorem 5.4. *The following transformation formulas for our extended Gauss and confluent hypergeometric functions holds true:*

$$F^{(\tau,k)}(a_1, a_2; a_3; t) = (1 - t)^{-a_1}F^{(\tau,k)}\left(a_1, a_3 - a_2; a_2; -\frac{t}{(1 - t)}\right) \tag{5.8}$$

$$(\tau \geq 0,\ \Re(k) > -1)$$

and

$$\Phi^{(\tau,k)}(a_2; a_3; t) = e^t\Phi^{(\tau,k)}(a_3 - a_2; a_3; -t) \tag{5.9}$$

$$(\tau \geq 0,\ \Re(k) > -1).$$

Proof. On replacing u by$1 - u$ in (5.3) and then by using $[1 - t(1 - u)]^{-a_1} = (1 - t)^{-a_1}\left[1 + \frac{t}{1-t}u\right]^{-a_1}$, we have

$$F^{(\tau,k)}(a_1, a_2; a_3; t) = \frac{(1 - t)^{-a_1}}{\mathbb{B}(a_2, a_3 - a_2)}$$

$$\times\int_0^1 u^{a_3 - a_2 - 1}\ (1 - u)^{a_2 - 1}\ \left(1 + \frac{t}{1 - t}u\right)^{-a_1} S_k\left[-\frac{\tau}{u(1 - u)}\right]du,$$

which further on using (5.3), yields the needed result (5.8). In a similar way, we can establish (5.9). □

Theorem 5.5. *The following summation formula for our extended Gauss hypergeometric function holds true:*

$$F^{(\tau,k)}(a_1, a_2; a_3; 1) = \frac{\mathbb{B}^{(\tau,k)}(a_2, a_3 - a_1 - a_2)}{\mathbb{B}(a_2, a_3 - a_2)} \tag{5.10}$$

$$(\tau \geq 0,\ \Re(k) > -1,\ \Re(a_3 - a_1 - a_2) > 0).$$

Proof. On setting $t = 1$ in (5.3) and then by using (2.1), we arrive at our desired result (5.10). □

Theorem 5.6. *The following generating function for our extended Gauss hypergeometric function holds true:*

$$\sum_{l=0}^{\infty}(a_1)_l\, F^{(\tau,k)}(a_1+l,a_2;a_3;t)\frac{u^l}{l!} = (1-u)^{-a_1}F^{(\tau,k)}\left(a_1,a_2;a_3;\frac{t}{1-u}\right) \tag{5.11}$$

$$(\tau \geq 0,\ \Re(k) > -1,\ |u| < 1).$$

Proof. On using (5.1) on the left side of (5.11), we have

$$\sum_{l=0}^{\infty}(a_1)_l\ F^{(\tau,k)}(a_1+l,a_2;a_3;t)\frac{u^l}{l!}$$

$$= \sum_{l=0}^{\infty}(a_1)_l\left[\sum_{m=0}^{\infty}\frac{(a_1+l)_m\ \mathbb{B}^{(\tau,k)}(a_2+m,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}\frac{t^m}{m!}\right]\frac{u^l}{l!}.$$

Now by using the identity $(a)_m(a+m)_l = (a)_l(a+l)_m$, in the above expression, we get

$$\sum_{l=0}^{\infty}(a_1)_l\ F^{(\tau,k)}(a_1+l,a_2;a_3;t)\frac{u^l}{l!}$$

$$= \sum_{m=0}^{\infty}(a_1)_m\frac{\mathbb{B}^{(\tau,k)}(a_2+m,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}\left[\sum_{l=0}^{\infty}(a_1+m)_l\frac{u^l}{l!}\right]\frac{t^m}{m!}.$$

$$= \sum_{m=0}^{\infty}(a_1)_m\frac{\mathbb{B}^{(\tau,k)}(a_2+m,a_3-a_2)}{\mathbb{B}(a_2,a_3-a_2)}(1-u)^{-(a_1+m)}\frac{t^m}{m!},$$

which, in view of (5.1), gives our needed result (5.11). □

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